Homotopy analysis method for fuzzy Black-Scholes equation

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Abstract In this work, the homotopy analysis method (HAM) is applied to solve the fuzzy Black-Scholes equation. Also, a theorem is proved to illustrate the convergence of the proposed method and two sample examples are solved by this method to verify the efficiency and importance of the method.

Keywords: Homotopy analysis method, Fuzzy Black-Scholes equation, Fuzzy numbers, convergence.

1 Introduction

In recent years, some numerical and analytical methods were proposed in order to solve fuzzy differential equations such as [1, 2, 3, 4, 5, 6, 7, 9, 11, 19, 20]. One of the powerful semi-analytical methods to solve differential equations is the homotopy analysis method. In [14, 15] the authors applied this method to solve the Schrodinger and Boussinesq equation in crisp case. In [8] the Black-Scholes equation in crisp case is solved by HAM. And in this work, we consider the fuzzy form of Black-Scholes equation as follows and apply the HAM to obtain the explicit series solution for it.

\[ \ddot{u} + \frac{\sigma^2 s^2}{2} \dddot{u} + R(t)s\dddot{u} - R(t)u = 0, \quad 0 \leq t \leq T, \quad s > 0. \]  

(1)

With the following initial condition,

\[ \ddot{u}(s, 0) = \dddot{f}(s). \]  

(2)

Where \( \ddot{u}(s, t) \) is the European call option price at asset price \( s \) and at time \( t \), \( T \) is the maturity, \( R(t) \) is the risk free interest rate, and \( \sigma(s, t) \) represents the volatility function of underlying asset.

In this case, we apply the homotopy analysis method in fuzzy case to find the solution of this equation. HAM is an important and efficient method to find the solution of the differential equations. The HAM, proposed
by Liao, [17, 18], is a semi analytical method which the solution is obtained as a series form according to a recursive relation stems from a deformation equation[14, 15].

In section 2, we remind some fuzzy concepts briefly. In section 3, we apply the HAM to solve the fuzzy Black-Sholes equation and we prove a theorem to show the convergence of the proposed method. In section 4, we solve two sample fuzzy Black-Sholes equations and we obtain a series solutions by this method, where it converges to the exact solution of the equations.

2 Preliminaries

In this section, we recall some basic definitions of fuzzy sets theory [23].

**Definition 2.1.** A fuzzy parametric number $u$ is a pair $(u(r), \pi(r))$, $0 \leq r \leq 1$, which satisfy the following requirements:

1. $u(r)$ is a bounded left continuous non-decreasing function over $[0, 1]$,
2. $\pi(r)$ is a bounded left continuous non-increasing function over $[0, 1]$,
3. $u(r) \leq \pi(r)$, $0 \leq r \leq 1$.

The set of all these fuzzy numbers is denoted by $E^1$. For $u = (u, \pi), v = (\tilde{u}, \tilde{\pi}) \in E^1, k \in \mathbb{R}$ the addition, multiplication and the scaler multiplication of fuzzy numbers are defined by

\[
(u + v)(r) = u(r) + v(r),
\]
\[
(\overline{u + v})(r) = \pi(r) + \tilde{\pi}(r),
\]
\[
(u \cdot v)(r) = \min\{u(r).v(r), u(r).\pi(r), v(r).\pi(r), \pi(r).\tilde{\pi}(r)\},
\]
\[
(\overline{u \cdot v})(r) = \max\{u(r).\overline{v}(r), u(r).\overline{\pi}(r), v(r).\overline{\pi}(r), \overline{\pi}(r).\overline{\tilde{\pi}}(r)\},
\]
\[
k_u(r) = k_u(r), \quad \overline{k_u}(r) = k_u(r), \quad k \geq 0,
\]
\[
k_u(r) = k_u(r), \quad \overline{k_u}(r) = k_u(r), \quad k \leq 0.\]

**Definition 2.2.** A fuzzy parametric number $\bar{u}$ is positive (negative) if and only if $u(r) \geq 0$ ($u(r) \leq 0$) $\forall r \in [0, 1]$.

**Remark 2.3.** If fuzzy parametric number $\bar{u}$ and $\bar{v}$ be positive, then $\bar{u} \cdot \bar{v}(r) = (u(r) \cdot v(r), \pi(r) \cdot \pi(r))$.

**Definition 2.4.** A function $f : \mathbb{R}^1 \rightarrow E^1$ is called a fuzzy function. If for arbitrary fixed $t_0 \in E^1$ and $\varepsilon > 0$ such that, $|t - t_0| < \delta \implies D(f(t), f(t_0)) < \varepsilon$ exists, $f$ is said to be continuous.[21, 22]

**Definition 2.5.** Let $u, v \in E^1$. If there exists $w \in E^1$ such that $u = v + w$, then $w$ is called the H-difference of $u, v$ and it is denoted by $u \ominus v.[10]$

**Definition 2.6.** Let $a, b \in \mathbb{R}$ and $f : (a, b) \rightarrow E^1$. Fix $t_0 \in (a, b)$. We say $F$ is strongly generalized differentiable at $t_0$, if there exists $f'(t_0) \in E^1$ such that
(i) for all $h > 0$ sufficiently close to 0, there exist $f(t_0 + h) \ominus f(t_0), f(t_0) \ominus f(t_0 - h)$ and the limits

\[
\lim_{h \to 0^+} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \to 0^+} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f'(t_0),
\]

or

(ii) for all $h > 0$ sufficiently close to 0, there exist $f(t_0 - h) \ominus f(t_0), f(t_0) \ominus f(t_0 + h)$ and the limits

\[
\lim_{h \to 0^+} \frac{f(t_0 - h) \ominus f(t_0)}{-h} = \lim_{h \to 0^+} \frac{f(t_0) \ominus f(t_0 + h)}{-h} = f'(t_0).
\]

or

(iii) for all $h > 0$ sufficiently close to 0, there exist $f(t_0 + h) \ominus f(t_0), f(t_0 - h) \ominus f(t_0)$ and the limits

\[
\lim_{h \to 0^+} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \to 0^+} \frac{f(t_0 - h) \ominus f(t_0)}{-h} = f'(t_0),
\]

or

(iv) for all $h > 0$ sufficiently close to 0, there exist $f(t_0) \ominus f(t_0 + h), f(t_0) \ominus f(t_0 - h)$ and the limits

\[
\lim_{h \to 0^+} \frac{f(t_0) \ominus f(t_0 + h)}{-h} = \lim_{h \to 0^+} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f'(t_0).
\]

($h$ and ($-h$) at denominators mean $\frac{1}{h}$ and $-\frac{1}{h}$ respectively).

Theorem 2.7. Let $f : (a, b) \to \mathbb{E}^1$ be strongly generalized differentiable on each point $t \in (a, b)$ in the sense of Definition 2.5, (iii) or (iv). Then $f'(x) \in \mathbb{R}$ for all $t \in (a, b)$ (see[10]).

Theorem 2.8. Let $f : \mathbb{R}^1 \to \mathbb{E}^1$ be a function and denote $f(t) = (\underline{f}(t, r), \overline{f}(t, r))$, for each $r \in [0, 1]$. Then (t) If $f$ is differentiable in the first form (i), then $\underline{f}(t, r)$ and $\overline{f}(t, r)$ are differentiable functions and $f'(t) = (\underline{f}'(t, r), \overline{f}'(t, r))$,

(2) If $f$ is differentiable in the second form (ii), then $\underline{f}(t, r)$ and $\overline{f}(t, r)$ are differentiable functions and $f'(t) = (\overline{f}'(t, r), \underline{f}'(t, r))$ (see[12]).

Definition 2.9. Let $a, b \in \mathbb{R}$ and $f : (a, b) \to \mathbb{E}^1$ and $t_0 \in (a, b)$. We define the $n$-th order differential of $f$ as follows: We say that $f$ is strongly generalized differentiable of $n$-th order at $t_0$, if there exists an element $f^{(s)}(t_0) \in \mathbb{E}^1 \forall s = 1, \ldots, n$ such that

(i) for all $h > 0$ sufficiently close to 0, there exist $f^{(s-1)}(t_0 + h) \ominus f^{(s-1)}(t_0), f^{(s-1)}(t_0) \ominus f^{(s-1)}(t_0 - h)$ and the limits

\[
\lim_{h \to 0^+} \frac{f^{(s-1)}(t_0 + h) \ominus f^{(s-1)}(t_0)}{h} = \lim_{h \to 0^+} \frac{f^{(s-1)}(t_0) \ominus f^{(s-1)}(t_0 - h)}{h} = f^{(s)}(t_0),
\]
For a given fuzzy function kind i or ii differentiability.

Remark 2.10. Note that by the above definition a fuzzy function is i-differentiable or ii-differentiable of order $n$. If $f(s)$ for $s = 1, \ldots, n$ is i-differentiable or ii-differentiable. It is possible that the different orders have different kind $i$ or $ii$ differentiability.

For a given fuzzy function $f$, we have two possibilities according to the definition 2.5 to obtain the derivative of $f$ at $t$: $D_1(f(t)), D_2(f(t))$. Then for each of these two derivatives, we have again two possibilities:

\[ D_1(D_1(f(t)) = D_1^{2.1}(f(t)), \quad D_2(D_1(f(t)) = D_2^{2.1}(f(t)), \]

\[ D_1(D_2(f(t)) = D_1^{2.2}(f(t)), \quad D_2(D_2(f(t)) = D_2^{2.2}(f(t)). \]

In similar, we can consider the n-order differential of $f$. For example $D_1^{3,1.1}(f(t)) = D_1(D_2(D_1(f(t))))$.

2.1 Main Idea

In order to describe the HAM for Eq.(1) we consider the following equation:

\[ \tilde{N}[\tilde{u}(s, t)] = \tilde{u}_t + \frac{\sigma^2 s^2}{2} \tilde{u}_{ss} + R(t)s\tilde{u}_s - R(t)\tilde{u} = 0, \]  

(3)
According to the parametric form of fuzzy numbers, we consider the Eq.(3) in the following form,

\[
N = \begin{pmatrix} N[u(s,t,r)] \\ N[u(s,t,r)] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad u(s,t,r) = \begin{pmatrix} u(s,t,r) \\ u(s,t,r) \end{pmatrix}.
\]

At first, we construct the zeroth-order deformation system.

\[
(I - Q)L[\phi(s,t,r;Q) - u_0(s,t,r)] = QHhN[\phi(s,t,r;Q)],
\]

where \(I\) is the identity matrix, \(L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}\) is an auxiliary linear operator matrix, \(H(x,t,r) = \begin{pmatrix} H_1(s,t,r) & 0 \\ 0 & H_2(s,t,r) \end{pmatrix}\) is an auxiliary function matrix, \(h = \begin{pmatrix} h_1 \\ 0 \\ 0 \\ h_2 \end{pmatrix}\) is an auxiliary parameter matrix, \(\phi(s,t,r;Q) = \begin{pmatrix} u(s,t,r) \\ u(s,t,r) \end{pmatrix}\) is an unknown function matrix, \(u_0(s,t,r)\) is an initial guess of the vector \(u(s,t,r)\) and \(Q = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\), \(0 \leq q \leq 1\), is a diagonal matrix which denotes the embedding parameter matrix. It is obvious, when the \(q\), increases from 0 to 1 or in other word the embedding parameter matrix changes from \(Q = 0\) to \(Q = I\), the solution of system of equations (4) changes from \(\phi(s,t,r;0) = u_0(s,t,r)\) to \(\phi(s,t,r;I) = u(s,t,r)\). Therefore, \(\phi(s,t,r)\) varies from the initial guess \(u_0(s,t,r)\) to the exact solution \(u(s,t,r)\) of the system.

We consider \(\phi(s,t,r;Q)\) in the following matrix expansion form,

\[
\phi(s,t,r;Q) = u_0(s,t,r) + \sum_{m=1}^{+\infty} Q^m u_m(s,t,r),
\]

where

\[
u_m(s,t,r) = \frac{1}{m!} \left( \frac{\partial^m \phi(s,t,r;q)}{\partial q^m} \bigg|_{q=0} \right).
\]

The convergence of the vector series (5) depends upon the auxiliary parameter matrix \(h\), if it is convergent at \(Q = I\), we have

\[
u(s,t,r) = u_0(s,t,r) + \sum_{m=1}^{+\infty} u_m(s,t,r).
\]

Now, we define the vectors,

\[
\vec{u}_k(s,t,r) = \{u_0(s,t,r), \ldots, u_k(s,t,r)\},
\]

where,
If we consider positive fuzzy number, there is no exception.

Proof. Without loss of generality, we suppose the series solution of problem (1) obtained from the HAM and also the series

where,

\[
\sum_{m=0}^{\infty} \frac{\partial^{m-1} N[\phi(s,t,r;Q)]}{\partial q^{m-1}} |Q=\overline{u}| = 0
\]

are convergent, and also it be a positive fuzzy number \( \forall t \in [0, T] \). Therefore we can write Eq.(1) in the following form,

\[
\bar{u}_t + \frac{\sigma^2 s^2}{2} \bar{u}_{ss} + [I(R(t)) + J(R(t)) ]u \bar{u}_{ss} - [I(R(t)) + J(R(t))]u = \bar{u}, \quad 0 \leq t \leq T, \quad s > 0.
\]

where,

\[
I(x) = \begin{cases} 
  x, & x \geq 0, \\
  0, & x < 0,
\end{cases}
J(x) = \begin{cases} 
  0, & x \geq 0, \\
  x, & x < 0.
\end{cases}
\]

Therefore, we have,

\[
\frac{N[u(s,t,r)]}{\overline{N}[u(s,t,r)]} = \begin{pmatrix} 
  \frac{\bar{u}_t + \frac{\sigma^2 s^2}{2} \bar{u}_{ss} + I(R(t)) \bar{u}_s + J(R(t)) \bar{u}_{ss} - I(R(t)) \bar{u} - J(R(t)) \bar{u} - 0}{\bar{u}_t + \frac{\sigma^2 s^2}{2} \bar{u}_s + I(R(t)) \bar{u}_s + J(R(t)) \bar{u}_s - I(R(t)) \bar{u} - J(R(t)) \bar{u} - 0} \\
  0 \\
\end{pmatrix}.
\]
If the series
\[\sum_{m=0}^{+\infty} u_m(s, t, r) = \left(\sum_{m=0}^{+\infty} u_m(s, t, r) \over \sum_{m=0}^{+\infty} \bar{u}_m(s, t, r)\right)\]
converges, we assume:
\[u(s, t, r) = \sum_{m=0}^{+\infty} u_m(s, t, r)\]
where
\[\lim_{m \to +\infty} u_m(s, t, r) = 0.\]  
(15)

We write
\[\sum_{m=1}^{n} [u_m(s, t, r) - \chi_m u_{m-1}(s, t, r)] = u_1 + (u_2 - u_1) + (u_3 - u_2) + \cdots + (u_n - u_{n-1}) = u_n(s, t, r),\]
using (15), we have,
\[\sum_{m=1}^{+\infty} [u_m(s, t, r) - \chi_m u_{m-1}(s, t, r)] = \lim_{n \to +\infty} u_n(s, t, r) = 0.\]

According to the definition of the operator \(L\), we can write
\[\sum_{m=1}^{+\infty} L[u_m(s, t, r) - \chi_m u_{m-1}(s, t, r)] = L \sum_{m=1}^{+\infty} [u_m(s, t, r) - \chi_m u_{m-1}(s, t, r)] = 0.\]

From above expression and equation (12), we obtain
\[\sum_{m=1}^{+\infty} L[u_m(s, t, r) - \chi_m u_{m-1}(s, t, r)] = hH(s, t) \sum_{m=1}^{+\infty} [R_m(\bar{u}_{m-1})].\]

Since \(h \neq 0\) and \(H(s, t) \neq 0\), we have
\[\sum_{m=1}^{+\infty} [R_m(\bar{u}_{m-1})] = 0.\]  
(16)

From (11), it holds
\[\sum_{m=1}^{+\infty} [R_m(\bar{u}_{m-1})] = \left\langle \frac{\partial}{\partial t} \sum_{m=1}^{+\infty} u_{m-1} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial s^2} \sum_{m=1}^{+\infty} u_{m-1} + I(R(t)) \frac{\partial}{\partial s} \sum_{m=1}^{+\infty} u_{m-1} + J(R(t)) \frac{\partial}{\partial s} \sum_{m=1}^{+\infty} \bar{u}_{m-1} \right.\]
\[-I(R(t)) \sum_{m=1}^{+\infty} \bar{u}_{m-1} - J(R(t)) \sum_{m=1}^{+\infty} \bar{u}_{m-1} - \sum_{m=1}^{+\infty} 0_{m-1} \left. \right)\].
Finally
\[ \sum_{m=1}^{+\infty} [R_m(\tilde{\mu}_m)] = \]
\[ \left( \frac{\partial}{\partial t} \sum_{m=0}^{+\infty} \tilde{u}_m + \frac{\sigma^2 s^2}{2} \sum_{m=0}^{+\infty} \tilde{\sigma}_m \partial_t \sum_{m=0}^{+\infty} \tilde{\sigma}_m + J(R(t)) \frac{\partial}{\partial s} \sum_{m=0}^{+\infty} \pi_m \right) \]
\[ - I(R(t)) \sum_{m=0}^{+\infty} \tilde{\pi}_m - J(R(t)) \sum_{m=0}^{+\infty} \tilde{\pi}_m - \sum_{m=0}^{+\infty} \tilde{\eta}_m \]
\[ \sum_{m=0}^{+\infty} \tilde{u}_m + \frac{\sigma^2 s^2}{2} \sum_{m=0}^{+\infty} \tilde{\sigma}_m + I(R(t)) \frac{\partial}{\partial s} \sum_{m=0}^{+\infty} \pi_m + J(R(t)) \frac{\partial}{\partial s} \sum_{m=0}^{+\infty} \pi_m \]
\[ = 0. \]

Therefore
\[ \begin{pmatrix} \tilde{u}_t + \frac{\sigma^2 s^2}{2} \tilde{u}_{ss} + I(R(t)) \tilde{u}_s + J(R(t)) \tilde{u}_s - I(R(t)) \tilde{u} - J(R(t)) \tilde{u} - 0 \\ \tilde{\pi}_t + \frac{\sigma^2 s^2}{2} \tilde{\pi}_{ss} + I(R(t)) \tilde{\pi}_s + J(R(t)) \tilde{\pi}_s - I(R(t)) \tilde{\pi} - J(R(t)) \tilde{\pi} - 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \]
and it means that,
\[ \tilde{u}_t + \frac{\sigma^2 s^2}{2} \tilde{u}_{ss} + R(t) \tilde{\pi}_{ss} - R(t) \tilde{u} = 0. \]

\[ \Box \]

3 Test Examples

Example 3.1. We consider the following fuzzy Black-Scholes equation
\[ \tilde{u}_t + \frac{1}{2} s^2 \tilde{u}_{ss} + s \tilde{u}_s - \tilde{u} = \tilde{0}, \]
with the initial conditions:
\[ \tilde{u}(s, 0) = (3r - 2, 2 - r) \frac{1}{\tilde{\sigma}}, \]
where \( \tilde{0} = (4r - 4, 4 - 4r)\tilde{u} \) and \( \tilde{u} \) is the solution of crisp case of the equation.

The exact solution of this equation is \( \tilde{u} = (3r - 2, 2 - r) \frac{1}{\tilde{\sigma}} \), therefore we can see \( \tilde{u}_t \) be i-differentiable with respect to the \( t \) and \( \tilde{u}, \tilde{\pi}, \tilde{\sigma} \) are also i-differentiable with respect to the \( s \). Hence we have
\[ \begin{pmatrix} \tilde{u}_t + \frac{1}{2} \tilde{s}^2 \tilde{u}_{ss} + \tilde{s} \tilde{u}_s - \tilde{u} - (4r - 4) \tilde{u} \\ \tilde{\pi}_t + \frac{1}{2} \tilde{s}^2 \tilde{\pi}_{ss} + \tilde{s} \tilde{\pi}_s - \tilde{\pi} - (4 - 4r) \tilde{\pi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

We consider \( H = I \) and \( h = -I \), and also, we choose the initial approximate as \( \begin{pmatrix} (3r - 2) \frac{1}{\tilde{\sigma}} \\ (2 - r) \frac{1}{\tilde{\pi}} \end{pmatrix} \). For using the Eq.(12), we have:
\[ R_m(\tilde{u}_m) = \begin{pmatrix} \frac{\partial}{\partial m} \tilde{u}_{m-1} + \frac{1}{2} \tilde{s}^2 I_{m-1}^{ss} \tilde{u}_{m-1} + s \frac{\partial}{\partial s} \tilde{u}_{m-1} - \tilde{\pi}_{m-1} - (4r - 4) \tilde{u}_{m-1} \\ \frac{\partial}{\partial m} \tilde{\pi}_{m-1} + \frac{1}{2} \tilde{s}^2 I_{m-1}^{ss} \tilde{\pi}_{m-1} + s \frac{\partial}{\partial s} \tilde{\pi}_{m-1} - \tilde{u}_{m-1} - (4 - 4r) \tilde{u}_{m-1} \end{pmatrix}. \]
Therefore,
\[
\begin{align*}
u_0 &= \left(\frac{3r-2}{s}, \frac{2-r}{s}\right) \\
u_1 &= \left(\frac{(3r-2)t}{s}, \frac{(2-r)t}{s}\right) \\
u_2 &= \left(\frac{(3r-2)t^2}{2s}, \frac{(2-r)t^2}{2s}\right)
\end{align*}
\]

In general the series solution is given by,
\[
u(s, t) = \sum_{n=0}^{\infty} u_n(s, t) = \left(\frac{(3r-2)(\frac{1}{s} + \frac{t^2}{2s^2} + \cdots)}{(2-r)(\frac{1}{s} + \frac{t^2}{2s^2} + \cdots)}\right).
\]

That gives the exact solution,
\[
\left(\frac{(3r-2)e^t}{s}, \frac{(2-r)e^t}{s}\right).
\]

Therefore \(\bar{u} = (3r - 2, 2 - r)e^t\) is the exact solution of the fuzzy equation.

**Example 3.2.** We consider the following fuzzy Black-Scholes equation
\[
\bar{u}_t + s^2 \bar{u}_{ss} + \frac{1}{2} s \bar{u}_s - \bar{u} = \bar{0},
\]

with the initial conditions:
\[
\bar{u}(s, 0) = \left(\frac{r}{3} + \frac{2}{3}\right) s^3 - 2r s^3,
\]

where \(\bar{0} = (17.5r - 17.5, 17.5 - 17.5r)\bar{u}\) and \(\bar{u}\) is the solution of crisp case of the equation.

The exact solution of this equation is \(\bar{u} = (\frac{r}{3} + \frac{2}{3}, 3 - 2r)s^3 e^{-6.5t}\), therefore we can see \(\bar{u}\) be ii-differentiable with respect to the \(t\) and \(\bar{u}, \bar{u}_s\) are i-differentiable with respect to the \(s\). Hence we have,
\[
\begin{pmatrix}
\bar{u}_t + s^2 \bar{u}_{ss} + \frac{1}{2} s \bar{u}_s - \bar{u} - (17.5r - 17.5)\bar{u} \\
\bar{u}_t + s^2 \bar{u}_{ss} + \frac{1}{2} s \bar{u}_s - \bar{u} - (17.5 - 17.5r)\bar{u}
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
We consider $H = I$ and $h = -I$, and also, we choose the initial approximate as $\begin{pmatrix} \left( \frac{5}{3} + \frac{s}{3} \right) s^3 \\ (3 - 2r) s^3 \end{pmatrix}$. For using the Eq.(12), we have:

$$R_m(\tilde{u}_{m-1}) = \begin{pmatrix} \frac{\partial}{\partial t} \tilde{u}_{m-1} + s^2 \frac{\partial^2}{\partial x^2} \tilde{u}_{m-1} + \frac{1}{2} \frac{s}{\partial x} \tilde{u}_{m-1} - \tilde{u}_{m-1} - (17.5 - 17.5r) \tilde{u}_{m-1} \\ \frac{\partial}{\partial t} \tilde{u}_{m-1} + s^2 \frac{\partial^2}{\partial x^2} \tilde{u}_{m-1} + \frac{1}{2} \frac{s}{\partial x} \tilde{u}_{m-1} - \tilde{u}_{m-1} - (17.5r - 17.5) \tilde{u}_{m-1} \end{pmatrix}.$$ 

Therefore,

$$u_0 = \begin{pmatrix} \left( \frac{5}{3} + \frac{s}{3} \right) s^3 \\ (3 - 2r) s^3 \end{pmatrix}$$

$$u_1 = \begin{pmatrix} -13s^3(2+r) \\ 13s^3(2r-3) \end{pmatrix}$$

$$u_2 = \begin{pmatrix} 169s^3t^2(2+r) \\ -169s^3t^2(2r-3) \end{pmatrix}$$

$$\vdots$$

In general the series solution is given by,

$$u(s, t) = \sum_{n=0}^{\infty} u_n(s, t) = \begin{pmatrix} (3r - 2)(\frac{1}{s} + \frac{1}{3} + \frac{t^2}{2s} + \cdots) \\ (2 - r)(\frac{1}{s} + \frac{1}{3} + \frac{t^2}{2s} + \cdots) \end{pmatrix}.$$ 

That gives the exact solution,

$$\begin{pmatrix} \left( \frac{5}{3} + \frac{2}{3} \right)e^{-6.5ts^3} \\ (3 - 2r)e^{-6.5ts^3} \end{pmatrix}.$$ 

Therefore $\bar{u} = \left( \frac{5}{3} + \frac{2}{3}, 3 - 2r \right)e^{-6.5ts^3}$ is the exact solution of the fuzzy equation.

4 Conclusion

In this work, we applied the fuzzy homotopy analysis method in order to solve the fuzzy Black-Sholes equation. For this aim, we considered the parametric form of a fuzzy number and established the deformation equations for two crisp equations obtained from the proposed method. Also, we presented a theorem to warrant the convergence of the proposed method too. Similar to the discussion in this work, the HAM can be used in order to solve other kinds of fuzzy differential equations as an efficient and proper method.
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