ON A SPECIAL FORM OF (V) HV-TORSION TENSOR $P_{ijk}$ IN FINSLER SPACES

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Abstract. In this present work, a special form of (v) hv-torsion tensor introduced which may be considered as a generalization of the $P$ - Finsler space and P-reducible Finsler space and then some properties of this space are studied.

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INTRODUCTION

Let $F^n (n \geq 3)$ be an n-dimensional Finsler space with metric function $L(x, y)$. There are five kinds of torsion tensors in the theory of Finsler space based on Cartan’s connection, out of which

$$P_{ijk} = y^h P_{hijk} \quad \text{and} \quad C_{ijk} = \frac{1}{4} \frac{\partial^3 L}{\partial y^i \partial y^j \partial y^k}$$

as (v)hv-torsion tensor and (h)hv-torsion tensor are of great importance tensors for the present study, where $P_{hijk}$ is as hv-curvature tensor. In Finsler geometry based on Cartan’s connection, there are three kinds of Covariant differentiations, out of which two of them are h-covarient differentiation denoted as $|_{i}$ and V-covarient differentiation denoted as $\tilde{i}$.

Various interesting forms of these tensors have been studied by many geometers ([1],[3],[4],[8],…). Two of them are a C-reducible Finsler space and a semi C-reducible Finsler space([6],[7]) in which the torsion tensor $C_{ijk}$ respectively is of the forms

$$C_{ijk} = \frac{1}{n+1} (C_i h_{jk} + C_j h_{ik} + C_k h_{ij}) ,$$

$$C_{ijk} = \frac{p}{n+1} (h_{ij} C_k + h_{ik} C_j + h_{jk} C_i) + \frac{q}{C^2} C_i C_j C_k$$

where $h_{ij}$ is the angular metric tensor and $C_i = C_{ik} g^{jk}$, where $g^{jk}$ is reciprocal of the metric tensor $g_{jk}$, and $p$ and $q$ are some scalar functions satisfying $p + q = 1$.

Izumi ([3],[4]) introduced $P^*$-Finsler space in which $P_{ijk}$ is of the form

$$P_{ijk} = \lambda C_{ijk} ,$$

where $\lambda$ is a scalar homogeneous function of degree zero in $y^i$. In a P-reducible Finsler space the tensor $P_{ijk}$ is of the form [9]
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\[ P_{ijk} = \frac{1}{n+1} (G_i h_{jk} + G_j h_{ik} + G_k h_{ij}) , \]

where \( G_i = C_{ij}^l y^l \). A Finsler space with \( P_{ijk} = 0 \) is called a Landsberg space [13]. If \( C_{ijkh} = 0 \), then \( F^n \) is called a Bewald’s affinely connected space ([2], [11]).

B. N. Prasad [11] introduced a special form of torsion tensor \( P_{ijk} \) as follow

\[ P_{ijk} = \lambda C_{ijk} + a_i h_{jk} + a_j h_{ik} + a_k h_{ij} , \]

where \( \lambda = \lambda(x, y) \) is a scalar homogenous function of degree 1 and \( a_i = a_i(x) \) is a homogenous function of degree 0 with respect to \( y^i \). He then studied some properties of \( F^n \) satisfying (5). The present author introduced a more general form of (5) and studied some properties of \( F^n \) satisfying it [14].

We quote the following lemmas, which will be used in the present paper.

**Lemma 1**: [6] If the hv-curvature tensor \( R_{ijk} \) of a C-reducible Finsler space vanishes then the space is Berwald’s affinely connected space.

**Lemma 2**: [5] A Finsler space \( F^n \) is locally Minkowskian iff \( h \)-curvature tensor \( R_{ijk} = 0 \), and \( |0| = h_{ijk} C \).

**Preliminaries**

Let \( M \) be an \( n \)-dimensional \( C^\infty \) manifold. By \( T_x M \) we mean the tangent space at \( x \in M \) and by \( TM \setminus \{0\} \) the slit tangent bundle of \( M \).

A Finsler metric on \( M \) is a function \( L : TM \rightarrow (0, \infty) \) which has the following properties:

\( i \) \( L \) is \( C^\infty \) on \( TM \setminus \{0\} \).

\( ii \) \( L \) is positively homogenous function of degree 1 on \( TM \).

\( iii \) For each \( y \in T_x M \), the metric tensor \( g_{ij} \), the angular metric tensor \( h_{ij} \) are respectively given by

\[ g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j} \quad \text{and} \quad h_{ij} = L \frac{\partial^2 L}{\partial y^i \partial y^j} . \]

The angular metric tensor \( h_{ij} \) can also be written in terms of the normalized element of support

\[ l_i = L g_{ij} y^j \quad \text{as} \quad h_{ij} = g_{ij} - l_i l_j \quad [7] . \]

For \( y \in T_x M \setminus \{0\} \), Cartan torsion tensor vector is defined as

\[ C_i := g^{ik} C_{ijk} . \]

According to Deicke’s theorem, \( C_i = 0 \) is the necessary and sufficient condition for \( F^n \) to be Riemannian.

Let \( F^n = (M^n, L) \) be a Finsler space. For \( y \in T_x M \setminus \{0\} \), we define Matsumoto torsions of C-reducible and Semi C-reducible Finsler spaces respectively as:

\[ M_{ijk} := C_{ijk} - \frac{1}{n+1} (C_i h_{jk} + C_j h_{ik} + C_k h_{ij}) \]

\[ \bar{M}_{ijk} := C_{ijk} - \frac{p}{n+1} (h_j C_k + h_k C_j + h_i C_r) - \frac{q}{C^2} C_i C_j C_k \]

A Finsler space \( F^n \) is said to be C-reducible if \( M_{ijk} = 0 \), and is Semi C-reducible if \( \bar{M}_{ijk} = 0 \).
Next, we define the tensor

\[ L_{ijk} := C_{ijk} y^j, \]

where the ‘|’ means h-covariant differentiation with respect to Cartan connection.

A Finsler space \( F^n \) is called a Landsberg space if \( P_{ijk} = 0 \), or equivalently \( L_{ijk} = C_{ijk} y^j = 0 \).

Define

\[ L_i := g^{jk} L_{ijk}. \]

A Finsler space \( F^n \) is said to be weakly Landsberg space if \( L_i = 0 \) [12].

It is obvious that every C-reducible Finsler space is P-reducible, but the converse is not true.

We define

\[ \tilde{M}_{ijk} := P_{ijk} - \frac{1}{n+1} (P_i h_{jk} + P_j h_{ik} + P_k h_{ij} ), \]

where

\[ P_{ijk} := \lambda C_{ijk} + \mu (a_i h_{jk} + a_j h_{ik} + a_k h_{ij}) + \nu C_i C_j C_k, \]

and

\[ P_i := g^{jk} P_{ijk}, \]

where \( \lambda \), \( \mu \) and \( \nu \) are some scalar function homogenous of degree 1, and \( a_i \)'s are homogenous of degree zero. It is obvious that \( F^n \) is a P-reducible Finsler space if \( \tilde{M}_{ijk} = 0 \). The purpose of the present paper is to study \( F^n \) satisfying (10).

If \( F^n \) is a Landsberg space then \( P_{ijk} = 0 \), therefore from (10) we get

\[ C_{ijk} = -\frac{\mu}{\lambda} (h_j a_k + h_k a_j + h_i a_j) - \frac{\nu}{\lambda} C_i C_j C_k, \]

and

\[ a_i = -\frac{p \lambda}{\mu (n+1)} C_i, \quad -\frac{\lambda C^2}{\mu} = q. \]

Hence we have the following

**Theorem 1.** A Landsberg space satisfying (10) is a semi C-reducible Finsler space.

Since for \( F^n \) to be a Landsberge space \( P_{ijk} = 0 \), therefore from Lemma 1 and Theorem 1,

**Theorem 2.** A Landsberg space satisfying (10) is a Berwald’s affinely connected space if \( \nu = 0 \).

In view of Lemma 2 and Theorem 2 we have the following

**Theorem 3.** If a Landsberg space satisfying (10) has vanishing h-curvature tensor i.e. \( R_{ijkh} = 0 \), then it is locally Minkowskian.

**Special form of** \( P_{ijk} \)

Let \( F^n \) be a Finsler space satisfying (10). A Finsler space with \( P_{ijk} \) of the given form reduces to a \( P^* \)-Finsler space when \( \mu = 0 = \nu \), while it reduces to a P-reducible Finsler space when \( \lambda = 0 = \nu \) and \( \mu a_i = \frac{1}{n+1} C_i \).

By definition, from (10) we can write

\[ L_{ijk} = \lambda C_{ijk} + \mu (a_i h_{jk} + a_j h_{ik} + a_k h_{ij}) + \nu C_i C_j C_k. \]
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Contracting above by \( g^{ij} \), we get

\[ L_k = (\lambda + \nu C^2)C_k + \mu(n+1)a_k, \]

or equivalently

\[ a_k = \frac{1}{\mu(n+1)}L_k - \frac{\lambda + \nu C^2}{\mu(n+1)}C_k. \]

By replacing (13) into (12), we obtain

\[ L_{ijk} = \lambda C_{ijk} + \frac{1}{n+1}(L_i h_{jk} + L_j h_{ik} + L_k h_{ij}) - \frac{\lambda + \nu C^2}{n+1}(C_i h_{jk} + C_j h_{ik} + C_k h_{ij}) + \nu C_i C_j C_k, \]

or

\[ L_{ijk} = \frac{1}{n+1}(L_i h_{jk} + L_j h_{ik} + L_k h_{ij}) = \lambda[C_{ijk} - \frac{p}{(n+1)}(C_i h_{jk} + C_j h_{ik} + C_k h_{ij}) - \frac{q}{C^2} C_i C_j C_k], \]

where \( p = 1 + \frac{\nu}{\lambda} C^2 \), and \( q = -\frac{\nu}{\lambda} C^2 \).

Hence we have the following

**Theorem 4.** The Matsumoto torsion of \( P \)-reducible Finsler space \( \tilde{M}_{ijk} \), and Matsumoto torsion of Semi \( C \)-reducible Finsler space \( \bar{M}_{ijk} \), are related by

\[ \tilde{M}_{ijk} = \lambda \bar{M}_{ijk}. \]

**Theorem 5.** A Finsler space \( F^n \) satisfying (10) is a weakly Landsberg space, if

\[ a_i = -\frac{\mu(n+1)}{\lambda + \nu C^2}V_i. \]

The proof immediately follows from contraction of (12) with \( g^{jk} \).

The notion of stretch curvature denoted by \( \Sigma_{hijk} \) was introduced by L. Berwald as a generalization of Landsberg curvature [2], in which

\[ \Sigma_{hijk} := 2(L_{hijk} - L_{hikj}). \]

A Finsler space \( F^n \) is said to be stretch space if \( \Sigma_{hijk} = 0 \).

Again taking h-covariant derivative of (12) and then contracting by \( y^h \), we get

\[ L_{ijk} y^h = (\lambda + \lambda^2)C_{ijk} + (\lambda \mu a_i + \nu a_i + \mu a_j + \mu a_j)h_{ik} + (\lambda \mu a_j + \mu a_j + \mu a_j)h_{ik} + (\lambda \mu a_k + \nu a_k) h_{ij} + (\nu C_k + V C) C_i C_j + (L_i C_j + L_j C) v C_k, \]

where we have put \( \lambda = \lambda_{jk}, \mu = \mu_{jk} \), and \( V = v_{jk} \).

Suppose that \( F^n \) be a stretch space, then

\[ L_{ijk} - L_{jik} = 0. \]

By contracting (14) with \( y^k \), we obtain

\[ L_{ijk} y^h = 0. \]

Putting (16) into (14), we have

\[ C_{ijk} = -\frac{1}{\lambda + \lambda^2}[(\lambda \mu a_i + \nu a_i + \mu a_j + \mu a_j)h_{ik} + (\lambda \mu a_j + \nu a_j + \mu a_j)h_{ik} + (\lambda \mu a_k + \nu a_k) h_{ij} + (\nu C_k + V C) C_i C_j + (L_i C_j + L_j C) v C_k], \]

[674]
Contraction of (17) by $g^{jk}$ yields

$$C_k = -\frac{1}{\lambda + \lambda^2} \left[ (n+1) (\lambda \mu a_k + \mu a_k + \mu a_k) + (\bar{\nu} C_k + \nu L_k) C^2 + 2\nu L C k \right].$$

Whence

$$\lambda \mu a_k + \mu a_k + \mu a_k = -\frac{\lambda + \lambda^2 + \nu C^2 + 2\nu L C}{n+1} C_k - \frac{\nu C^2}{n+1} L k.$$  

Substituting (18) into (17), we get

$$C_{ijk} = \frac{\beta + \lambda^2 + \nu C^2 + 2\nu L C}{(n+1)(\lambda + \lambda^2)} (C_i h_{jk} + C_j h_{ik} + C_k h_{ij}) + \frac{\nu C^2}{(n+1)(\lambda + \lambda^2)} (L_i h_{jk} + L_j h_{ik} + L_k h_{ij})$$

$$+ \left( \frac{-\nu}{\lambda + \lambda^2} \right) C_i C_j C_k + \left( \frac{-\nu}{\lambda + \lambda^2} \right) (L_i C_j C_k + C_i L_j C_k + C_i C_j L_k).$$  

Form (19), it follows that $F^n$ is a Semi C-reducible Finsler space if it is a weakly Landsberg space.

Therefore we have the following

**Theorem 6.** Let a Finsler space $F^n$ satisfying (10) be a stretch space, then it is a Semi C-reducible Finsler space, if it is a weakly Landsberg space.

**REFERENCES**


