G- Brownian motion and Its Applications

Atena EBRAHIMBEYG1, Elham DASTRANJ2

1 M.Sc. Student, Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Shahrood, Shahrood, Iran

2 Academic Member, Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Shahrood, Shahrood, Iran

Received: 01.02.2015; Accepted: 05.05.2015

Abstract. The concept of G-Brownian motion and G-Ito integral has been introduced by Peng. Also Ito isometry lemma is proved for Ito integral and Brownian motion. In this paper we first investigate the Ito isometry lemma for G-Brownian motion and G-Ito Integral. Then after studying of $M_{G}^{2,0}$-class functions [4], we introduce Stratonovich integral for G-Brownian motion, say G-Stratonovich integral. Then we present a special construction for G-Stratonovich integral.

Keywords: G-expectation, G-Brownian motion, Characterization, Ito integral, G-Stratonovich.

1. INTRODUCTION

The concept of G-Brownian motion is a very important concept in financial mathematics. With G-Brownian motion, G-Ito integral for $M_{G}^{2,0}$-class function has been introduced in [2,3,4,5]. In this paper we introduce G-Stratonovich integral for $M_{G}^{2,0}$-class functions. In the sequel we present a characterization for G-Stratonovich in integral which we define.

2. NONLINEAR EXPECTATIONS

Let $\Omega$ be a given set and let $\mathcal{H}$ be a linear space of real functions defined on $\Omega$ containing 1, namely $\mathcal{H}$ is a linear space such that $1 \in \mathcal{H}$ and that $X \in \mathcal{H}$ implies $|X| \in \mathcal{H}$. $\mathcal{H}$ is a space of random variables. We assume the functions on $\mathcal{H}$ are all bounded.

Definition 2.1. [4] A non linear expectation $\mathbb{E}$ is a functional $\mathcal{H} \rightarrow \mathbb{R}$ satisfying the following properties

a) Monotonicity: if $X, Y \in \mathcal{H}$ and $X \geq Y$ then $\mathbb{E}[X] \geq \mathbb{E}[Y]$,

b) Preservation of constants: $\mathbb{E}[c] = c$,

c) Subadditivity $\mathbb{E}[X] - \mathbb{E}[Y] = \mathbb{E}[X-Y], \forall X, Y \in \mathcal{H}$,

d) Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \forall \lambda \geq 0, X \in \mathcal{H}$.

e) $\mathbb{E}[X + c] = \mathbb{E}[X] + c$.

*Corresponding author. Email address: atena.ebrahimi42@gmail.com

Special Issue: The Second National Conference on Applied Research in Science and Technology

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3. G-NORMAL DISTRIBUTIONS

For a given positive integer n, we will denote by $(x, y)$ the scalar product of $x, y \in \mathbb{R}^n$ and by $|x|=(x, x)^{1/2}$ the Euclidean norm of $x$. We denote by $\text{lip}(\mathbb{R}^n)$ the space of all bounded and Lipschitz real functions on $\mathbb{R}^n$. We introduce the notion of nonlinear distribution—G–normal distribution. A G–normal distribution is a nonlinear expectation defined on $\text{lip}(\mathbb{R}^d)$ (here $\mathbb{R}^d$ is considered as $\Omega$ and $\text{lip}(\mathbb{R}^d)$ as $\mathcal{H}$):

$$P^G_1(\emptyset) = u(1,0) : \emptyset \in \text{lip}(\mathbb{R}^d) \rightarrow \mathbb{R}$$

where $u = u(t, x)$ is a bounded continuous function on $[0, \infty) \times \mathbb{R}^d$ which is the viscosity solution of the following nonlinear parabolic partial differential equation (PDE)

$$\frac{du}{dt}G(D^2 u) = 0 , \quad u(0, x) = \emptyset(x) , \quad (t, x) \in [0, \infty) \times \mathbb{R}^d ,$$

(1)

here $D^2 u$ is the Hessian matrix of $u$, i.e., $D^2 u = (\partial^2_{x^i x^j} u)_{i, j = 1}^d$ and

$$G(A) = G_\tau(A) = \frac{1}{2} \sup \text{tr}[\gamma \gamma^T A] , \quad A = (A_{ij})_{i, j = 1}^d \in \mathbb{S}_d$$

(2)

$\mathbb{S}_d$ denotes the space of $d \times d$ symmetric matrices. $\tau$ is a given non empty, bounded and closed subset of $\mathbb{R}^{d \times d}$, the space of all $d \times d$ matrices.

3.1. Dimensional G-Brownian motion under G-expectation

In this section we use some definitions and notions of [2,4].

Let $\Omega = C_0(\mathbb{R}^+)$ be the space of all $\mathbb{R}$-valued continuous paths $(\omega_t)_{t \in \mathbb{R}^+}$ with $\omega_0 = 0$. For any $\omega^1, \omega^2 \in \Omega$ we define

$$\rho(\omega^1, \omega^2) = \sum_{i=1}^{\infty} 2^{-i} \left[ \left( \max_{t \in [0, 1]} |\omega^1_t - \omega^2_t| \right) \wedge 1 \right].$$

We set, for each $t \in [0, \infty)$

$$\mathcal{W}_t := \{ \omega_{s \wedge t} : \omega \in \Omega \},$$

$$\mathcal{F}_t := \mathcal{B}(\mathcal{W}_t),$$

$$\mathcal{F}_{t+} := \mathcal{B}(\mathcal{W}_t^+) = \mathcal{B} \left( \mathcal{W}_{t+} \right),$$

$$\mathcal{F} := \bigvee_{t \geq 0} \mathcal{F}_t$$

Then $(\Omega, \mathcal{F})$ is the canonical space with the natural filtration. This space is used throughout the rest of this paper.

For each fixed $T \geq 0$, we consider the following space of random variables
Obviously, it holds \( l^p(\mathcal{F}_t) \subseteq l^p(\mathcal{F}_T) \), for any \( t \leq T < \infty \). We further define,
\[
l^p(\mathcal{F}) := \bigcap_{n=1}^{\infty} l^p(\mathcal{F}_n).
\]
We will consider the canonical space and set \( B_t(\omega) = \omega_t, \ t \in [0,\infty) \), for \( \omega \in \Omega \).

**Definition 3.1.** The canonical process \( GB \) is called a (d–dimensional) G–Brownian motion under a nonlinear expectation \( \mathbb{E} \) defined on \( L^p_0(\mathcal{F}) \) if

(i) For each \( s,t \geq 0 \) and \( \psi \in \text{lip}(\mathbb{R}^d) \), \( GB_t \) and \( GB_{t+s} - GB_s \) are identically distributed:
\[
\mathbb{E}[\psi(GB_{t+s} - GB_s)] = \mathbb{E}[\psi(GB_t)].
\]

(ii) For each \( m = 1,2, \ldots, 0 \leq t_1 < \cdots < t_m < \infty \), the increment \( GB_{t_m} - GB_{t_{m-1}} \) is “backwardly” independent from \( GB_{t_{m-1}} - GB_{t_{m-2}} \) in the following sense: for each \( \varnothing \in \text{lip}(\mathbb{R}^{d \times m}) \),
\[
\mathbb{E}[\varnothing(GB_{t_1}, \ldots, GB_{t_m})] = \mathbb{E}[\varnothing(GB_{t_1}, \ldots, GB_{t_{m-1}}, GB_{t_m})]
\]
where \( \varnothing(x^1, \ldots, x^{m-1}) = \mathbb{E}[\varnothing(x^1, \ldots, x^{m-1}, GB_{t_m} - GB_{t_{m-1}} + x^{m-1})] \), \( x^1, \ldots, x^{m-1} \in \mathbb{R}^d \).

The related conditional expectation of \( \varnothing(GB_{t_1}, \ldots, GB_{t_m}) \) under \( \mathcal{F}_{t_k} \) is defined by
\[
\mathbb{E}[\varnothing(GB_{t_1}, \ldots, GB_{t_k}, \ldots, GB_{t_m})|\mathcal{F}_{t_k}] = \varnothing_{m-k}(GB_{t_1}, \ldots, GB_{t_k})
\]
where
\[
\varnothing_{m-k}(x^1, \ldots, x^{k}) = \mathbb{E}[\varnothing(x^1, \ldots, x^{k}, GB_{t_{k+1}} - GB_{t_k} + x^k, \ldots, GB_{t_m} - GB_{t_k} + x^k)].
\]

**Definition 3.2.**
\[
M^0_\omega(0,T) := \{\eta_\xi : (\omega) = \xi \left[ \sum_{j=1}^{n-1} \xi_j I_{[t_j, t_{j+1})}(t) \right], \ \forall n > 0, 0 \leq t_0 \leq \cdots \leq t_m, \ \xi_j(\omega) \in L_0(\mathcal{F}_{t_j}), i = 0, \ldots, n-1\}[^4].
\]

**Definition 3.3.** [1] In the sequel we assume \( (\Omega, \mathcal{F}, \mathbb{P}) \) is a fixed probability space. \( f(t,\omega) : [0,\infty) \times \Omega \rightarrow \mathbb{R} \) is belongs to \( P_2 = P_2(S,T) \) Class functions set if and only if we have,

(i) \((t,\omega) \rightarrow f(t,\omega)\) is \( \mathcal{B} \times \mathcal{F}\)-measurable, where \( \mathcal{B} \) denotes the Borel \( \sigma \)-field on \([0,\infty)\).

(ii) For \( t \in [0,\infty) \), \( f(t,.) \) is \( \mathcal{F}_t \)-adapted.

(iii) \( \mathbb{E}[\int^T_T f(t,\omega) dt < \infty \ \forall T \geq 0] \).

**Remark 3.1.** (The Itô isomery)[1] let \( \varnothing(t,\omega) \in P_2 \) be bounded and elementary function,

Then we have
\[
\mathbb{E}[\int^T_T \varnothing(t,\omega) dB_t] = \mathbb{E}[\int^T_T \varnothing(t,\omega)^2 dt],
\]
where \( \int^T_T \varnothing(t,\omega) \ dB_t \) is Itô intgral [1].
Remark 3.2. The isometry lemma for G-Brownian motion is not necessary holded i.e.

There is \( \eta \in M^2 (0, T) \) such that,

\[
E \left[ \left( \int_0^T \eta(S) \, dGB \right)^2 \right] \neq E \int_0^T \eta(S)^2 \, dGB
\]

4. G-STRATONOVICH (STRATONOVICH INTEGRAL FOR G-BROWNIAN MOTION)

Definition 4.1. For \( T \in \mathbb{R}_+ \), a partition \( \rho \) of \( [0, T] \) is a finite ordered subset \( \rho = \{t_0, ..., t_n\} \) such that \( 0 = t_0 < t_1 < \cdots < t_n = T \).

\[
\mu(\rho) = \max(|t_{i+1} - t_i|, i = 0, 1, ..., N - 1)
\]

We use \( \mathcal{P}_T^N = \{t_0^N < t_1^N < \cdots < t_N^N\} \) to denote a sequence of partitions of \( [0, T] \) such that \( \lim_{N \to \infty} \mu(\mathcal{P}_T^N) = 0 \).

For each \( f \in M^2 (0, T) \)

We denote G-Stratonovich integral as following

\[
\int_0^T f(t, \omega) \, d(GB) = \lim_{N \to \infty} \sum_{i=1}^{N} f(t^*, \omega) \langle GB \rangle_{t_i} - \langle GB \rangle_{t_{i-1}}
\]

Where \( t^* = \frac{t_j - t_{j-1}}{2} \).

In the following theorem we present a characterization for G-Stratonovich integral.

Theorem 4.1. In the above definition if we choose \( t^* \) randomly with the Uniform distribution then the random sequence tends to G-Stratonovich integral when \( n \) tends to \( \infty \).

Proof. If we choose \( t_i^* \)’s randomly with the Uniform distribution and show the resulting integral with

\[
\mathbb{U}^* \int_0^T f(t, \omega) \, d(GB),
\]

then it’s not difficult to show that

\[
E(\mathbb{U}^* \int_0^T f(t, \omega) \, d(GB) - \int_0^T f(t, \omega) \, d(GB) | t_i^* \langle GB \rangle)
\]

tends to zero, where \( \mathbb{U} \) is defined in definition 3.1.

5. CONCLUSION

For G- Brownian motion, Stratonovich integral which we call it G-Stratonovich integral is definable. Also we presented a random characterization for G-Stratonovich integral.

ACKNOWLEDGMENT
The research of authors was supported financially by University of Shahrood. So we should thank the University of Shahrood for its financially support.

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