SPLIT SEMI-QUATERNIONS ALGEBRA IN SEMI-EUCLIDEAN 4-SPACE

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ABSTRACT. The aim of this paper is to study the split semi-quaternions, $H_{ss}$, and to give some of their algebraic properties. We show that the set of unit split semi-quaternions is a subgroup of $H_{ss}^0$. Furthermore, with the aid of De Moivre’s formula, any powers of these quaternions can be obtained.

Keywords De Moivre’s formula, Split semi-quaternion, Euler’s formula.

1. Introduction

Quaternions were invented by Sir William Rowan Hamilton as an extension to the complex number in 1843. Hamilton’s defining relation is most succinctly written as

$$i^2 = j^2 = k^2 = ijk = -1.$$  

Quaternions have provided a successful and elegant means for the representation of three dimensional rotations, Lorentz transformations of special relativity, robotics, computer vision, problems of electrical engineering and so on. The Euler’s and De-Moivre’s formulas for the complex numbers are generalized for quaternions. Obtaining the roots of a quaternion was given by Niven[3] and Brand [1]. Brand proved De Moivre’s theorem and used it to find n-th roots of a quaternion. These formulas are also investigated in the cases of split and semi-quaternions [2, 4]. A brief introduction of the split semi-quaternions is provided in [5]. In this paper, we investigate some algebraic properties of split semi-quaternions. Moreover, we obtain De-Moivre’s and Euler’s formulas for these quaternions in different cases. We use De-Moivre’s formula to find $n-$th roots of a split semi-quaternion. Finally, we give some example for the purpose of more clarification.

2. Splitsemi-quaternions

Definition 2.1. A split semi-quaternion $q$ is defined as

$$q = a_0 + a_1i + a_2j + a_3k$$

where $a_0, a_1, a_2$ and $a_3$ are real numbers and $i, j, k$ are quaternionic units with the properties that

$$i^2 = 1, \quad j^2 = k^2 = 0$$

$$ij = k = -ji, \quad jk = 0 = kj$$

and

$$ki = -j = -ik.$$
The set of all split semi-quaternions are denoted by \( H_{ss} \). A split semi-quaternion \( q \) is a sum of a scalar and a vector, called scalar part, \( S_q = a_0 \), and vector part \( V_q = a_1i + a_2j + a_3k \). The set of split semi-quaternions \( H_{ss} - \{ [0, (0,0,0)] \} \) is written \( H_{ss}^o \).

Let \( q, p \in H_{ss} \), where \( q = S_q + V_q \) and \( p = S_p + V_p \). The addition operator, +, is defined

\[
q + p = (S_q + S_p) + (V_q + V_p) = (a_0 + b_0) + (a_1 + b_1)i + (a_2 + b_2)j + (a_3 + b_3)k.
\]

This rule preserves the associativity and commutativity properties of addition. The product of scalar and a split semi-quaternion is defined in a straightforward manner. If \( c \) is a scaler and \( q \in H_{ss} \),

\[
cq = cS_q + cV_q = (ca_0)1 + (ca_1)i + (ca_2)j + (ca_3)k.
\]

The multiplication rule for split semi-quaternions is defined as

\[
qp = S_qS_p - <V_q, V_p> + S_qV_p + S_pV_q + V_q \times V_p,
\]

where

\[
<V_q, V_p> = -a_1b_1, V_p \times V_q = 0i - (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k.
\]

It could be written as

\[
qp = \begin{bmatrix}
a_0 & a_1 & 0 & 0 \\
a_1 & a_0 & 0 & 0 \\
a_2 & -a_3 & a_0 & a_1 \\
a_3 & -a_2 & a_1 & a_0
\end{bmatrix}
\begin{bmatrix}
b_0 \\
b_1 \\
b_2 \\
b_3
\end{bmatrix}.
\]

Split semi-quaternion multiplication is not generally commutative. We state the following properties of quaternion multiplication:

**Proposition 2.1.** Let \( q, q', p \in H_{ss} \) and \( r \in \mathbb{R} \). Then

\[
(pq)q' = p(qq') \quad (Quatertion multiplication is associative.)
\]

\[
p(q + q') = pq + pq' \quad (Quatertion multiplication distributes across addition.)
\]

**Corollary 2.1.** \( H_{ss} \) with addition and multiplication has all the properties of a number field except commutativity of the multiplication. It is therefore called the skew field of quaternions.
3. Some properties of split semi-quaternions

**Definition 3.1.** Let \( q \in H_{ss} \). Then \( \overline{q} \) is called the conjugate of \( q \) if it is defined by
\[
\overline{q} = a_0 - (a_1 i + a_2 j + a_3 k) = S_q - V_q.
\]

It is clear the scalar and vector part of \( q \) is denoted by \( S_q = \frac{a_0 + \overline{a}_1}{2} \) and \( V_q = \frac{a_0 - \overline{a}_1}{2} \).

The above definition would lead to the following properties:

**Proposition 3.1.** Let \( q, p \in H_{ss} \). Then

1. \( \overline{q} = q \)
2. \( pq = qp \)
3. \( q + p = q + p \)
4. \( qq = qq \).

**Definition 3.2.** Let \( q \in H_{ss} \) and let the mapping \( \| \cdot \| : H_{ss} \to \mathbb{R} \) be defined by \( \| q \| = a_0^2 - a_1^2 \in \mathbb{R} \). This mapping is called the norm and \( \| q \| (= N_q) \) is norm of \( q \). If \( \| q \| = a_0^2 - a_1^2 = 1 \), then \( q \) is called a unit split semi-quaternion. We will use \( H_{ss}^1 \) to denote the set of unit split semi-quaternions.

A split semi-quaternion \( q \) for which \( \| q \| = 0 \) has the form \( q = a_2 j + a_3 k \) \((a_0 = a_1 = 0)\) and it is a zero divisor, but not all zero divisors of this algebra have this form.

**Definition 3.3.** Let \( q \in H_{ss} \) and \( \| q \| \neq 0 \). Then there exists \( q^{-1} \in H_{ss} \) such that \( qq^{-1} = q^{-1} q = 1 \). Furthermore \( q^{-1} \) is unique and it is given by
\[
q^{-1} = \frac{\overline{q}}{\| q \|}.
\]

**Proposition 3.2.** Let \( p, q \in H_{ss} \) and \( \lambda \in \mathbb{R} \). The following three equations hold:

1. \( (qp)^{-1} = p^{-1} q^{-1} \)
2. \( (\lambda q)^{-1} = \frac{1}{\lambda} q^{-1} \)
3. \( \| q^{-1} \| = \frac{1}{\| q \|} \).

**Proposition 3.3.** The set \( H_{ss}^1 \) of unit split semi-quaternions is a subgroup of the group \( H_{ss}^2 \).

**Proof.** Let \( q, q' \in H_{ss}^1 \). We have \( \| q q' \| = 1 \), i.e. \( qq' \in H_{ss}^1 \) and thus the first subgroup requirement is satisfied. Also, by proposition 3.2,
\[
\| q \| = \| \overline{q} \| = \| q^{-1} \| = 1.
\]

and thereby the second subgroup requirement \( q^{-1} \in H_{ss}^1 \).

4) To divide a split semi-quaternion \( p \) by the semi-quaternion \( q(N_q \neq 0) \), one simply has to resolve the equation
\[
xq = p \quad \text{or} \quad qy = p,
\]
with the respective solutions
\[
x = pq^{-1} = p \frac{\overline{q}}{N_q},
\]
\[
y = q^{-1}p = \frac{\overline{q}}{N_q}.
\]
and the relation $N_x = N_y = \frac{N_z}{N_v}$.

**Definition 3.4.** Let $q, p \in H_{ss}$, $q = S_q + V_q$ and $p = S_p + V_p$. The inner product is defined as

$$g(q, p) = S_q S_p + <V_q, V_p> = S(qp).$$

**Theorem 3.1.** The inner product has the properties;

1) $g(pq_1, pq_2) = N_p \cdot g(q_1, q_2)$
2) $g(q_1 p, q_2 p) = N_p \cdot g(q_1, q_2)$
3) $g(pq_1, q_2) = g(q_1, q_2)$
4) $g(pq_1, q_2) = g(p, q_2 q_1)$.

**Proof.** We will prove the identities (1) and (3).

\begin{align*}
    g(pq_1, pq_2) &= S(pq_1 \overline{pq_2}) = S(pq_1 \overline{q_2} p) \\
               &= S(q_2 \overline{p} pq_1) = N_p S(q_2 q_1) \\
               &= N_p S(q_1 q_2) = N_p \cdot g(q_1, q_2),
\end{align*}

and

\begin{align*}
    g(pq_1, q_2) &= S(pq_1 \overline{q_2}) = S(q_1 \overline{q_2} p) \\
               &= S(q_1 \overline{p} q_2) = g(q_1, pq_2).
\end{align*}

**Theorem 3.2.** The algebra $H_{ss}$ is isomorphic to the subalgebra of the algebra $\mathbb{C}_2'$ consisting of the $(2 \times 2)$-matrices

$$\hat{A} = \begin{bmatrix} A & B \\ 0 & \overline{A} \end{bmatrix},$$

and to the subalgebra of the algebra $\mathbb{C}_2$ consisting of the $(2 \times 2)$-matrices

$$\hat{A} = \begin{bmatrix} A & B \\ 0 & A \end{bmatrix},$$

where $A, B \in \mathbb{C}$.

**Proof.** The proof can be found in [5].

4. De Moivre's formula for split semi-quaternions

In this section, we express De-Moivre’s formula for split semi-quaternions. For this, we can consider two different cases:

**Case 1:** Let the norm of split semi-quaternion be positive.
Definition 4.1. Every nonzero split semi-quaternion \( q = a_0 + a_1 i + a_2 j + a_3 k \) can be written in the polar form

\[
q = r (\cosh \varphi + \vec{w} \sinh \varphi)
\]

where \( r = \sqrt{N_q} \) and

\[
\cosh \varphi = \frac{|a_0|}{r}, \quad \sinh \varphi = \frac{|a_1|}{\sqrt{a_0^2 - a_1^2}}.
\]

The unit vector \( \vec{w} \) is given by

\[
\vec{w} = \frac{1}{\sqrt{a_1^2}} (a_1 i + a_2 j + a_3 k), \quad a_1 \neq 0.
\]

Euler’s formula for a unit split semi-quaternion holds. Since \( \vec{w} \vec{w} = 1 \), we have

\[
e^{\vec{w} \varphi} = 1 + \vec{w} \varphi + \frac{(\vec{w} \varphi)^2}{2!} + \frac{(\vec{w} \varphi)^3}{3!} + \ldots
\]

\[
= (1 + \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} + \ldots) + \vec{w} (\varphi + \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} + \ldots)
\]

\[
= \cosh \varphi + \vec{w} \sinh \varphi.
\]

Moreover, this can be shown by using the following method.

\[
q = \cosh \varphi + \vec{w} \sinh \varphi \Rightarrow dq = (\sinh \varphi + \vec{w} \cosh \varphi) d\varphi
\]

\[
dq = \vec{w} (\cosh \varphi + \vec{w} \sinh \varphi) d\varphi = \vec{w} q d\varphi.
\]

thus, we get

\[
\int \frac{dq}{q} = \int \vec{w} d\varphi \Rightarrow \ln q = \vec{w} \varphi \Rightarrow q = e^{\vec{w} \varphi} = \cosh \varphi + \vec{w} \sinh \varphi.
\]

Example 4.1. The polar form of split semi-quaternions \( q_1 = 2 + \sqrt{2} i + \sqrt{2} k, q_2 = 3 + 2 i + 3 j + k, q_3 = 4 + i + 2 j + 2 k \) are

\[
q_1 = \sqrt{2} (\cosh \theta_1 + \vec{w} \sinh \theta_1) \quad \text{where} \theta_1 = \ln(\sqrt{2} + 1),
\]

\[
q_2 = \sqrt{5} (\cosh \theta_2 + \vec{w} \sinh \theta_2) \quad \text{where} \theta_2 = \ln(\sqrt{5}), \quad \text{and}
\]

\[
q_3 = \sqrt{15} (\cosh \theta_3 + \vec{w} \sinh \theta_3) \quad \text{where} \theta_3 = \ln(\frac{5}{\sqrt{15}}), \text{respectively}.
\]

Lemma 4.1. Let \( \vec{w} \) be a unit vector, then we have

\[
(\cosh \varphi + \vec{w} \sinh \varphi)(\cosh \psi + \vec{w} \sinh \psi) = \cosh(\varphi + \psi) + \vec{w} \sinh(\varphi + \psi).
\]

Now, let’s prove De Moivre’s formula for a split semi-quaternion.

Theorem 4.1. (De-Moivre’s formula) Let \( q = \cosh \varphi + \vec{w} \sinh \varphi \) be a unit split semi-quaternion. Then for every integer \( n \):

\[
q^n = \cosh n \varphi + \vec{w} \sinh n \varphi.
\]

Proof. We use induction on positive integers \( n \). Assume that \( q^n = \cosh n \varphi + \vec{w} \sinh n \varphi \) holds. Then

\[
q^{n+1} = (\cosh \varphi + \vec{w} \sinh \varphi)^n (\cosh \varphi + \vec{w} \sinh \varphi)
\]

\[
= (\cosh n \varphi + \vec{w} \sinh n \varphi) (\cosh \varphi + \vec{w} \sinh \varphi)
\]

\[
= \cosh(n \varphi + \varphi) + \vec{w} \sinh(n \varphi + \varphi)
\]

\[
= \cosh(n+1) \varphi + \vec{w} \sinh(n+1) \varphi.
\]
The formula holds for all integer $n$, since

\[
q^{-1} = \cosh \varphi - \overrightarrow{w} \sinh \varphi,
\]

\[
q^{-n} = \cosh(-n\varphi) + \overrightarrow{w} \sinh(-n\varphi)
= \cosh n\varphi - \overrightarrow{w} \sinh n\varphi.
\]

**Example 4.2.** Let $q = 3 - 2i - j + 3k$ be a split semi-quaternion. Then, we can write it as $q = \sqrt{5}(\cosh \theta + \overrightarrow{w} \sinh \theta)$ where $\theta = \ln(\sqrt{5})$. Every power of this quaternion is found with the aid of Theorem 4.2, for example, 10-th power of

\[
q^{10} = 5^5[\cosh 10\theta + \overrightarrow{w} \sinh 10\theta],
\]

where $\cosh 10\theta = \frac{5^5 + 5^{-5}}{2}$ and $\sinh 10\theta = \frac{5^5 - 5^{-5}}{2}$.

**Theorem 4.2.** Let $q = \cosh \varphi + \overrightarrow{w} \sinh \varphi$ be a unit split semi-quaternion. The equation $x^n = q$ has only one root:

\[
x = \cosh \frac{\varphi}{n} + \overrightarrow{w} \sinh \frac{\varphi}{n}.
\]

**Proof.** If $x^n = q$, $q$ will have the same unit vector as $\overrightarrow{w}$. So, we assume that $x = \cosh \chi + \overrightarrow{w} \sinh \chi$ is a root of the equation $x^n = q$. From Theorem 4.2, we have

\[
x^n = \cosh n\chi + \overrightarrow{w} \sinh n\chi,
\]

Thus, $\chi = \frac{\varphi}{n}$. So, $x = \cosh \frac{\varphi}{n} + \overrightarrow{w} \sinh \frac{\varphi}{n}$ is a root of the equation $x^n = q$.

**Example 4.3.** Let $q = 2 + \sqrt{3}i - 2j + k = (\cosh \varphi + \overrightarrow{w} \sinh \varphi)$ be a split semi-quaternion. The equation $x^3 = q$ has one root and that is

\[
x = (\cosh \frac{\ln(2 + \sqrt{3})}{3} + \overrightarrow{w} \sinh \frac{\ln(2 + \sqrt{3})}{3}).
\]

**Case 2:** Let the norm of split semi-quaternion be negative.

**Definition 4.2.** Every nonzero split semi-quaternion $q = a_0 + a_1i + a_2j + a_3k$ can be written in the polar form

\[
q = r(\sinh \psi + \overrightarrow{u} \cosh \psi)
\]

where $r = \sqrt{|N_q|}$ and

\[
\sinh \psi = \frac{|a_0|}{r}, \quad \cosh \psi = \frac{\sqrt{a_1^2}}{r} = \frac{|a_1|}{\sqrt{|a_0^2 - a_1^2|}}.
\]

The unit vector $\overrightarrow{u}$ is given by

\[
\overrightarrow{u} = \frac{1}{\sqrt{a_1^2}}(a_1i + a_2j + a_3k), \quad a_1 \neq 0.
\]
Example 4.4. The polar form of the split semi-quaternions $q_1 = 2 + 3i - j + 2k$, $q_2 = 1 + \sqrt{2}i + 2j + k$ are $q_1 = \sqrt{5}(\sinh \theta_1 + \sqrt{u} \cosh \theta_1)$ where $\theta_1 = \ln \sqrt{5}$, $q_2 = \sinh \theta_2 + \sqrt{u} \cosh \theta_2$ where $\theta_2 = \ln(1 + \sqrt{2})$, respectively.

**Theorem 4.3.** (De-Moivre’s formula) Let $q = \sinh \varphi + \sqrt{u} \cosh \varphi$ be a unit split semi-quaternion. Then for every integer $n$:

$$q^n = \sinh n\varphi + \sqrt{u} \cosh n\varphi.$$ 

Example 4.5. Let $q = \sqrt{2} + 2i - j + 3k = \sqrt{2}(\sinh \theta + \sqrt{u} \cosh \theta)$ be a split semi-quaternion. Every power of this split semi-quaternion is found by the aid of Theorem 4.4, for example, 40-th power is

$$q^{40} = 2^{20}[\sinh 40\theta + \sqrt{u} \cosh 40\theta],$$

where $\sinh 40\theta = \frac{(1+\sqrt{2})^{40}-(1-\sqrt{2})^{-40}}{2}$ and $\cosh 40\theta = \frac{(1+\sqrt{2})^{40}+(1-\sqrt{2})^{-40}}{2}$.

**Theorem 4.4.** Let $q = \sinh \varphi + \sqrt{u} \cosh \varphi$ be a unit split semi-quaternion. The equation $x^n = q$ has only one root:

$$x = \sinh \frac{\varphi}{n} + \sqrt{u} \cosh \frac{\varphi}{n}.$$

**Proof.** If $x^n = q$, $q$ will have the same unit vector as $\sqrt{u}$. So, we assume that $x = \sinh \theta + \sqrt{u} \cosh \theta$ is a root of the equation $x^n = q$. From Theorem 4.4, we have

$$x^n = \cosh n\chi + \sqrt{u} \sinh n\chi,$$

Thus, $\theta = \frac{\varphi}{n}$. So, $x = \sinh \frac{\varphi}{n} + \sqrt{u} \cosh \frac{\varphi}{n}$ is a root of the equation $x^n = q$.

Example 4.6. Let $q = 2 + 3i - 2j + k = (\sinh \varphi + \sqrt{u} \cosh \varphi)$ be a split semi-quaternion. The equation $x^4 = q$ has one root and that is

$$x = (\sinh \frac{\ln(\sqrt{5})}{4} + \sqrt{u} \cosh \frac{\ln(\sqrt{5})}{4}).$$

5. Conclusion

In this paper, we give some of algebraic properties of the split semi-quaternions and investigate the Euler’s and De Moivre’s formulas for these quaternions in different cases. We use it to find $n$–th roots of a split semi-quaternion.
References


